Graph Theory

Definition. *A **graph** G consists of a finite set of elements V called **vertices**, and a finite set of elements E called **edges** and a function $E \to V \times V$ called the edge-endpoint function. The number of vertices and edges is denoted by $|V|$ and $|E|$, respectively.*

In principle it is possible for a graph to have multiple edges connecting a pair of vertices, and edges which connect the same the vertex (that is, which loop at the vertex). However, we will here exclude the possibility of multiple edges or loops.

Definition. *Two vertices are said to be **adjacent** if they are joined by an edge. Suppose that we have enumerated the vertices of a graph from 1 to n. Then the graph is uniquely associated with an **adjacency matrix**. This is the $n \times n$ matrix whose entry in the i^{th} row and j^{th} column is 1 if the i^{th} vertex is joined to the j^{th} vertex, and 0 otherwise. If vertices are joined by $k > 1$ edges, then the corresponding entry is k.*

Definition. *A **directed graph** or **digraph** has directional edges, so that having an edge from vertices v_1 to v_2 does not imply there is an edge from v_2 to v_1. You should think of the edges in a directed graph as arrows between vertices (with the possibility that the arrows might be double-headed).*

Definition. *Let v be the vertex of a graph. The **degree** of v, denoted $\deg(v)$ is defined to be the number of edges which have v as an endpoint.*

Lemma (Handshaking Lemma).

$$\sum_{v \in V} \deg(v) = 2|E|$$

Definition. *Two graphs G and H are said to be **isomorphic** if there exists a one-to-one correspondence between their vertex sets such that the number of edges joining any two vertices in G is equal to the number of edges joining the corresponding vertices in H. Such a mapping is referred to as an isomorphism between G and H.*

Note that the number of possible graphs on a given set of n vertices is $2^{\binom{n}{2}}$. However, the number of non-isomorphic graphs on a given set of n vertices is much fewer.

Definition. *A **subgraph** of G is a graph whose sets of vertices and edges are subsets of the sets of vertices and edges of G.*

Definition. *Suppose we have a graph G.*

*(i) A **walk** is an alternating sequence $v_1, e_1, v_2, e_2 \dots v_k, e_k$ of vertices and edges, where the edge e_i joins the vertices v_i and v_{i+1}. It is sufficient to just list the vertices (unless the graph contains multiple edges connecting a pair of vertices).*

*(ii) A **trail** is a walk which does not use the same edge twice.*

*(iii) A **path** is a walk which does not use the same vertex twice (and, hence, doesn't use the same edge twice).*

Note that every path is a trail, and every trail is a walk.

Definition. *We say a graph is **connected** if there is a path between any two vertices, and say a graph is **disconnected** if it is not connected.*

Definition. *Suppose we have a graph G.*

*(i) A walk is said to be **closed** if it begins and ends at the same vertex.*

*(ii) A **circuit** is a closed trail.*

*(iii) A **cycle** is a closed path (that is, it begins and ends at the same vertex, but otherwise doesn't use any vertex twice or any edge twice) containing at least one edge.*

Definition. *Suppose we have a graph G.*

*(i) A walk that uses every edge of the graph exactly once is called an **Euler trail**.*

*(ii) A walk that uses every edge of the graph exactly once and which starts and ends at the same vertex is called an **Euler circuit**.*

The following theorem gives a succinct characterization for when an Euler circuit exists.

Theorem. *A connected graph has an Euler circuit iff every vertex has even degree.*

Definition. *Suppose we have a graph G.*

*(i) A walk that uses every vertex of the graph exactly once is called an **Hamiltonian path**.*
*(ii) A walk that uses every vertex of the graph exactly once and which starts and ends at the same vertex is called a **Hamiltonian cycle**.*

Unlike for Euler circuits, there is no easily checkable necessary and sufficient condition for a graph to have a Hamiltonian cycle.

Definition. *We say that a walk has **length** k if it involves traversing k edges.*

Proposition. *If A is the adjacency matrix of a graph G, then the entry in the i^{th} row and j^{th} column of A^k gives the number of walks of length k from vertex v_i to v_j.*

Definition. *A **regular graph** is a graph for which all vertices have the same degree. If a regular graph has n vertices each of degree r, then it must have $nr/2$ edges. Two special examples of regular graphs are **cycle graphs** and **complete graphs**. The cycle graph with n vertices is denoted C_n and every vertex has degree 2. The complete graph with n vertices is denoted K_n and every vertex has degree $n-1$.*

Definition. *The **complement graph** of a graph G is another graph denoted by $\bar{G}$ and has the same vertices as G. Two vertices in $\bar{G}$ are joined iff they are not joined in G.*

Definition. *A **bipartite graph** is a graph whose set of vertices can be divided into two disjoint subsets S_1 and S_2, such that the edges do not join vertices within either of these subsets. A **complete bipartite graph** is a bipartite graph in which every vertex in S_1 is joined to every vertex in S_2. The complete bipartite graph in which S_1 has m vertices and S_2 has n vertices, is denoted $K_{m,n}$. It has $m+n$ vertices and mn edges.*

Definition. *An edge $e \in E$ of a graph G is said to be a **bridge** of G if $G-e$ has more connected components than G. In particular, an edge e in a connected graph G is a bridge iff $G-e$ is disconnected.*

Definition. *A **tree** is a connected graph that contains no cycles. In general, a graph (not necessarily connected) without cycles is called **acyclic**.*

Note that every tree is a bipartite graph.

Theorem. *Suppose G is a connected graph. Then the following are equivalent*

(i) G is a tree.
(ii) Any two vertices are connected by a unique path.
(iii) $|E| = |V| - 1$.
(iv) Every edge is a bridge.

Definition. *Suppose G is a connected graph. A **spanning tree** of G is a subgraph of G that contains all the vertices of G, and is a tree.*

Theorem. *Every connected graph G has a spanning tree. Moreover, for any connected graph $|E| \geq |V| - 1$.*

Definition. *A graph G is called **planar** if it can be drawn in the plane in such a way that no edges meet, except at vertices which they both have as an endpoint. Every plane drawing of G divides the plane into regions called **faces** (including one face of infinite extent). Let F denote the set of faces of a graph and $|F|$ denote the number of faces.*

Theorem (Euler's Formula). *Let G be a connected, planar graph. Then,*

$$|V| - |E| + |F| = 2.$$

Theorem. *If G is a planar graph G with $|V| \geq 3$, then*

$$|E| \leq 3|V| - 6.$$

Definition. *A graph H is said to be a **subdivision** of a graph G if H is obtained from G by introducing one or more new vertices along the edges of G.*

Theorem (Kuratowski). *A graph is planar iff it contains no subdivision of K_5 or $K_{3,3}$ as a subgraph.*

The above theorem says that K_5 and $K_{3,3}$ are non-planar graphs, and that they are essentially the "smallest" non-planar graphs, in the sense that every non-planar graph contains a copy of at least one of them.

Definition. *A **weighted graph** is a graph equipped with a function $\alpha : E \to \mathbb{R}$ which assigns a weighting to each of the edges. This concept is prevalent in applications where the weighting may be thought of as the distance between vertices. Given a weighted graph we define the distance between $v_1, v_2 \in V$ as follows*

$$d(v_1, v_2) = \min \left\{ \sum_{1 \le i \le k} \alpha(e_i) : e_1 e_2 \dots e_k \textit{ is a path from } v_1 \textit{ to } v_2 \right\}$$